

CHARACTERISATION OF L_p -NORMS VIA HÖLDER'S INEQUALITY

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ABSTRACT. We characterise L_p -norms on the space of integrable step functions, defined on a probabilistic space, via Hölder's type inequality with an optimality condition.

1. INTRODUCTION

In a series of papers Matkowski ([4], [5], [6], [7], [8]), jointly with Świątkowski ([9], [10]), derived several characterisations of the L_p -norm via classical Hölder's and Minkowski's inequalities. In this paper we will deal with a certain, in a sense critical, case concerning the Hölder inequality.

Hereinafter (X, Σ, μ) stands for a measure space. Every μ -integrable function will be treated as an element of $L_1(\mu)$, *i.e.*, we interpret equality between such functions in the μ -almost everywhere sense. We denote $\mathcal{S} = \mathcal{S}(X)$ the vector space of all Σ -integrable step functions and $\mathcal{S}_+ = \mathcal{S}_+(X) = \{f \in \mathcal{S} : f \geq 0\}$. Denote $\mathbb{R}_+ = [0, \infty)$ and for any bijection $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ with $\varphi(0) = 0$ set

$$\mathbb{P}_\varphi(f) = \varphi^{-1} \left(\int_X \varphi \circ f \, d\mu \right) \quad \text{for } f \in \mathcal{S}.$$

We are motivated by the following result of Matkowski:

Theorem 1 ([6, Theorem 3]). *Suppose that there are two sets $A, B \in \Sigma$ such that*

$$(1) \quad 0 < \mu(A) < 1 < \mu(B) < \infty$$

and $\varphi, \psi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are bijections satisfying $\varphi(0) = \psi(0) = 0$ and

$$(2) \quad \int_X fg \, d\mu \leq \mathbb{P}_\varphi(f) \mathbb{P}_\psi(g) \quad \text{for } f, g \in \mathcal{S}_+.$$

Then there exist numbers $p, q > 1$ with $p^{-1} + q^{-1} = 1$ such that $\varphi(t) = \varphi(1)t^p$ and $\psi(t) = \psi(1)t^q$ for $t \in \mathbb{R}_+$.

It was also shown in [6] that assumption (1) in the above theorem is essential. Namely, we have what follows:

Theorem 2 ([6, Theorem 5]). *Suppose that (X, Σ, μ) is a probabilistic space (*i.e.* $\mu(X) = 1$) such that for at least one set $A \in \Sigma$ we have $0 < \mu(A) < 1$. Then, bijections $\varphi, \psi : \mathbb{R}_+ \rightarrow$*

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\mathbb{R}_+ with $\varphi(0) = \psi(0) = 0$ satisfy inequality (2) if and only if the map $F: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ defined by $F(s, t) = \varphi^{-1}(s)\psi^{-1}(t)$ is concave.

As we see, the condition $\mu(X) = 1$ is critical and in this case Hölder's inequality does not determine a concrete form of φ and ψ . The aim of this paper is to show that this situation may be fixed by adopting additionally the following consistency condition:

(*) For every non-zero function $f \in \mathcal{S}_+$ there exists a function $\chi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\chi(0) = 0$ and $\chi \circ f \neq 0$ and such that inequality (2) becomes equality for $g = \chi \circ f$. Conversely, for every non-zero $g \in \mathcal{S}_+$ there is a function $\tau: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\tau(0) = 0$ and $\tau \circ g \neq 0$ and such that inequality (2) becomes equality for $f = \tau \circ g$.

It simply says that inequality (2) is assumed to be optimal for any given map from \mathcal{S} . Recall that the equality case in Hölder's inequality for exponents p and q occurs exactly when f^p and g^q are proportional, hence the above condition holds true with $\chi(t) = t^{p/q}$ and $\tau(t) = t^{q/p}$. That is why, we believe, it is a natural requirement.

Our idea is based on introducing a new measure space (Y, T, ν) , with total mass greater than 1, and then applying a result (Theorem 3) concerning generalised Minkowski's inequality for the product space $(X \times Y, \Sigma \otimes T, \mu \otimes \nu)$. Let us recall that the mentioned inequality asserts that if μ and ν are σ -finite measures and $F: X \times Y \rightarrow \mathbb{R}_+$ is $(\Sigma \otimes T)$ -measurable, then for every $1 \leq p < \infty$ we have

$$(3) \quad \left\{ \int_X \left(\int_Y F(x, y) \nu(dy) \right)^p \mu(dx) \right\}^{1/p} \leq \int_Y \left(\int_X F(x, y)^p \mu(dx) \right)^{1/p} \nu(dy).$$

2. RESULTS

Theorem 3. *Let (X, Σ, μ) and (Y, T, ν) be measure spaces such that $[0, 1] \subset \mu(\Sigma)$ and $[0, \alpha] \subset \nu(T)$ for some $\alpha > 1$. Suppose $\varphi, \psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfy $\varphi(0) = \psi(0) = 0$ and $\varphi(\mathbb{R}_+) = [0, \beta]$ (may be either right-closed or right-open) for some $\beta \in (0, \infty]$. Then, the inequality*

$$(4) \quad \psi \left\{ \int_X \varphi \left(\int_Y F(x, y) \nu(dy) \right) \mu(dx) \right\} \leq \int_Y \psi \left(\int_X \varphi \circ F(x, y) \mu(dx) \right) \nu(dy)$$

holds true for every $F \in \mathcal{S}_+(X \times Y)$ if, and only if, either:

- (i) $\psi|_{\varphi(\mathbb{R}_+)} = 0$, or
- (ii) $\varphi(1) \neq 0 \neq \psi(1)$ and there exists $p \geq 1$ such that $\varphi(t) = \varphi(1)t^p$ and $\psi(t) = \psi(1)t^{1/p}$ for $t \in \mathbb{R}_+$.

Proof. For arbitrary $A_1 \in \Sigma$ and $B_1 \in T$, and any $t \in \mathbb{R}_+$, we have

$$\psi(\mu(A_1)\varphi(\nu(B_1)t)) \leq \nu(B_1)\psi(\mu(A_1)\varphi(t)),$$

which follows from (4) by taking $F = t\mathbb{1}_{A_1 \times B_1}$. Hence,

$$\psi(a\varphi(bt)) \leq b\psi(a\varphi(t)) \quad \text{for } a \in [0, 1], b \in [0, \alpha] \text{ and } t \in \mathbb{R}_+.$$

Therefore, if $b, t > 0$ then we have

$$\frac{\psi(a\varphi(bt))}{bt} \leq \frac{\psi(a\varphi(t))}{t},$$

whence substituting $v = bt$ (then $v/t = b \in [0, \alpha]$) gives

$$(5) \quad \frac{\psi(a\varphi(v))}{v} \leq \frac{\psi(a\varphi(t))}{t} \quad \text{for } a \in [0, 1] \text{ and } t, v > 0 \text{ such that } \frac{v}{t} \leq \alpha.$$

For any $\delta > 0$ and $a \in [0, 1]$ define a map $\Phi_{\delta,a}: [\delta, \alpha\delta] \rightarrow \mathbb{R}_+$ by

$$\Phi_{\delta,a}(x) = \frac{\psi(a\varphi(x))}{x}.$$

Since $x/y \leq \alpha$ for every x and y from $[\delta, \alpha\delta]$, inequality (5) implies that $\Phi_{\delta,a}$ is constant. Hence, there is $M(\delta, a) \in \mathbb{R}_+$ satisfying

$$(6) \quad \psi(a\varphi(x)) = M(\delta, a)x \quad \text{for } a \in [0, 1], \delta > 0 \text{ and } x \in [\delta, \alpha\delta].$$

Taking any sequence $(\delta_n)_{n=-\infty}^{\infty}$ of positive numbers with

$$\delta_{-n} \xrightarrow{n \rightarrow \infty} 0, \quad \delta_n \xrightarrow{n \rightarrow \infty} 0 \quad \text{and} \quad \delta_n < \delta_{n+1} < \alpha\delta_n \text{ for } n \in \mathbb{Z},$$

we can see that the numbers $M(\delta, a)$ in equation (6) do not depend on δ . Thus, there is a function $M: [0, 1] \rightarrow \mathbb{R}_+$ such that

$$(7) \quad \psi(a\varphi(x)) = M(a)x \quad \text{for } a \in [0, 1] \text{ and } x \in \mathbb{R}_+.$$

If $M(a) = 0$ for some $a \in (0, 1]$, then (7) would imply that ψ vanishes on the interval $\varphi(\mathbb{R}_+)$, that is, assertion (i) holds true. So, for the rest of the proof we may assume that $M(a) \neq 0$ for every $a \in (0, 1]$.

Let $a, b \in [0, 1]$. Composing the two functions: $x \mapsto \psi(a\varphi(x))$ and $x \mapsto \psi(b\varphi(x))$, and using (7), we obtain

$$\psi(a\varphi(\psi(b\varphi(x)))) = M(a)M(b)x \quad \text{for } x \in \mathbb{R}_+.$$

Fixing for a moment the variable x and regarding the both sides of the above equation as functions of a and b , we conclude by symmetry that

$$(8) \quad \psi(a\varphi(\psi(b\varphi(x)))) = \psi(b\varphi(\psi(a\varphi(x)))).$$

Now, observe that ψ is a one-to-one function on the set $\{a\varphi(x): a \in [0, 1], x \in \mathbb{R}_+\}$. For if $0 \leq a \leq b \leq 1$ and $x, y \in \mathbb{R}_+$ satisfy $a\varphi(x) \neq b\varphi(y)$, then $a/b \leq 1$, so $a/b \cdot \varphi(x) \in \varphi(\mathbb{R}_+)$, say $\varphi(z) = a/b \cdot \varphi(x)$. Of course, $b \neq 0$, so $M(b) \neq 0$. Thus,

$$\psi(a\varphi(x)) = \psi(b\varphi(z)) = M(b)z \neq M(b)y = \psi(b\varphi(y)).$$

Consequently, equation (8) implies that $a\varphi(\psi(b\varphi(x))) = b\varphi(\psi(a\varphi(x)))$, that is,

$$\frac{\varphi \circ \psi(a\varphi(x))}{a} = \frac{\varphi \circ \psi(b\varphi(x))}{b} \quad \text{for } a, b \in (0, 1] \text{ and } x \in \mathbb{R}_+.$$

Hence, there is a map $\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying

$$\varphi \circ \psi(a\varphi(x)) = a\gamma(x) \quad \text{for } a \in [0, 1] \text{ and } x \in \mathbb{R}_+.$$

Now, for any $a, b \in [0, 1]$ and $x \in \mathbb{R}_+$ we have

$$M(a)M(b)x = \psi(b\varphi(\psi(a\varphi(x)))) = \psi(ab\gamma(x)),$$

thus there is a map $N: [0, 1] \rightarrow \mathbb{R}_+$ such that $\psi(c\gamma(x)) = N(c)x$ for every $c \in [0, 1]$, $x \in \mathbb{R}_+$ and, moreover,

$$N(ab) = M(a)M(b) \quad \text{and} \quad N(a) = M(1)M(a) \quad \text{for } a, b \in [0, 1].$$

Therefore, the function $m: (0, 1] \rightarrow \mathbb{R}_+$ given by $m(x) = M(x)/M(1)$ is multiplicative (*i.e.* satisfies $m(xy) = m(x)m(y)$) and does not vanish, which implies that

$$m(x) = (\exp \circ A \circ \log)(x) \quad \text{for } x \in (0, 1],$$

where $A: \mathbb{R} \rightarrow \mathbb{R}$ is an additive function ([3, Theorem 13.1.3]).

Using equation (7) we get

$$(9) \quad \psi(a\varphi(x)) = M(a)x = M(1)m(a)x = M(1)(\exp \circ A \circ \log)(a)x$$

for $a \in (0, 1]$ and $x \in \mathbb{R}_+$. Putting here $x = 1$ yields

$$\psi(\varphi(1)a) = M(1)(\exp \circ A \circ \log)(a) \quad \text{for } a \in (0, 1]$$

(which, in particular, implies $\varphi(1) \neq 0$), thus

$$(10) \quad \psi(z) = M(1)(\exp \circ A \circ \log)\left(\frac{z}{\varphi(1)}\right) \quad \text{for } z \in (0, \varphi(1)].$$

Fix any $x > 0$ and pick $a \in (0, 1]$ such that $a\varphi(x) \leq \varphi(1)$. Applying (10) to $z = a\varphi(x)$ we obtain

$$\psi(a\varphi(x)) = M(1)(\exp \circ A \circ \log)\left(a\frac{\varphi(x)}{\varphi(1)}\right).$$

On the other hand, we have formula (9), and consequently,

$$A\left(\log a\frac{\varphi(x)}{\varphi(1)}\right) = A(\log a) + \log x,$$

that is

$$A(\log \varphi(x)) = A(\log \varphi(1)) + \log x \quad \text{for } x \in (0, \infty).$$

The last equation implies that A is surjective and also injective on the interval $(-\infty, \log \beta|]$ (as $x \in (0, \infty)$ the values $\varphi(x)$ runs through $(0, \beta|]$). This implies that A is injective on the whole real line (otherwise for some h_1, \dots, h_k from a Hamel basis of \mathbb{R} over \mathbb{Q} , and some $\lambda_1, \dots, \lambda_k \in \mathbb{Q}$, not all equal to zero, we would have $\sum_j \lambda_j f(h_j) = 0$, but then $f(k \sum_j \lambda_j h_j) = 0$ for every $k \in \mathbb{Q}$, so f would vanish at infinitely many points of $(-\infty, \log \beta|]$). Applying the function $\exp \circ A^{-1}$ to the both sides of the equation above we get

$$\varphi(x) = \varphi(1) \exp\{A^{-1}(\log x)\} \quad \text{for } x \in (0, \infty)$$

and, coming back to (9), we get

$$\psi(x) = \frac{M(1)}{\varphi(1)} \exp\{A(\log x)\} \quad \text{for } x \in (0, \infty).$$

Replacing, with no loss of generality, φ by $\varphi/\varphi(1)$ and ψ by $\psi/\psi(1)$ we may conclude that both φ and ψ are multiplicative on $(0, \infty)$ and $\psi = \varphi^{-1}$.

Now, pick any sets $A_1, A_2 \in \Sigma$ with $A_1 \cap A_2 = \emptyset$ and $B_1, B_2 \in T$ with $B_1 \cap B_2 = \emptyset$ such that

$$b := \mu(A_1) = \mu(A_2) > 0 \quad \text{and} \quad c := \nu(B_1) = \nu(B_2) > 0$$

(we may take $b = c = 1/2$). For arbitrary $t, u, v, w \in \mathbb{R}_+$ define a map $F \in \mathcal{S}_+$ by

$$F(x, y) = \begin{cases} G(x) & \text{if } y \in B_1, \\ H(x) & \text{if } y \in B_2, \\ 0 & \text{if } y \in Y \setminus (B_1 \cup B_2), \end{cases}$$

where

$$G = t\mathbb{1}_{A_1} + u\mathbb{1}_{A_2} \quad \text{and} \quad H = v\mathbb{1}_{A_1} + w\mathbb{1}_{A_2}.$$

Plugging this function into inequality (4), and using multiplicativity of φ , we obtain

$$\begin{aligned} \varphi^{-1} \left\{ \int_X \varphi \left(\int_Y F(x, y) \nu(dy) \right) \mu(dx) \right\} &= \varphi^{-1} \left\{ \int_X \varphi(cG(x) + cH(x)) \mu(dx) \right\} \\ &= c\varphi^{-1} \left\{ \int_X \varphi(G(x) + H(x)) \mu(dx) \right\} = c\varphi^{-1} \left(\int_{A_1} + \int_{A_2} \right) \\ &= c\varphi^{-1}(b\varphi(t + v) + b\varphi(u + w)) \leq \int_Y \varphi^{-1} \left(\int_X \varphi \circ F(x, y) \mu(dx) \right) \nu(dy) \\ &= \int_{B_1} \varphi^{-1} \left(\int_X \varphi(G(x)) \mu(dx) \right) \nu(dy) + \int_{B_2} \varphi^{-1} \left(\int_X \varphi(H(x)) \mu(dx) \right) \nu(dy) \\ &= c\varphi^{-1} \left(\int_X \varphi(G(x)) \mu(dx) \right) + c\varphi^{-1} \left(\int_X \varphi(H(x)) \mu(dx) \right) \\ &= c\varphi^{-1}(b\varphi(t) + b\varphi(u)) + c\varphi^{-1}(b\varphi(v) + b\varphi(w)). \end{aligned}$$

Therefore, dividing by $\varphi^{-1}(b)$ and c , and defining a function $\mathbf{p}_\varphi: \mathbb{R}_+^2 \rightarrow \mathbb{R}$ by the formula

$$\mathbf{p}_\varphi(\mathbf{t}) = \varphi^{-1}(\varphi(t_1) + \varphi(t_2)) \quad \text{for } \mathbf{t} = (t_1, t_2) \in \mathbb{R}_+^2,$$

we may write

$$\mathbf{p}_\varphi(t + v, u + w) \leq \mathbf{p}_\varphi(t, u) + \mathbf{p}_\varphi(v, w) \quad \text{for } t, u, v, w \in \mathbb{R}_+$$

(the Mulholland inequality; see [11] and [3, §8.8]). By appealing to Theorem 4 below, due to Matkowski and Świątkowski, we infer that φ is a convex homeomorphism, whence $\varphi(t) = t^p$ for some $p \geq 1$, and all $t \in \mathbb{R}_+$. Consequently, $\psi(t) = t^{1/p}$ for all $t \in \mathbb{R}_+$ and the proof is completed. \square

Theorem 4 ([10, Theorem 2]). *If $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bijection and the function \mathbf{p}_φ , defined as above, is subadditive on \mathbb{R}_+^2 , then φ is a convex homeomorphism of \mathbb{R}_+ .*

We may invert inequality (4) and, with obvious changes in the proof of Theorem 3, derive the multiplicativity of φ and ψ , and the relation $\psi = \varphi^{-1}$. Next, we may conclude that the map \mathbf{p}_φ is superadditive, hence, by an analogue of Theorem 4 ([10, Theorem 3]), the function φ is a concave homeomorphism. Therefore, we obtain the following counterpart of Theorem 3:

Theorem 5. *Let (X, Σ, μ) , (Y, T, ν) , φ and ψ be as in Theorem 3, but instead of (4) assume the reversed inequality. Then, and only then, we have either:*

- (i) $\psi|_{\varphi(\mathbb{R}_+)} = 0$, or
- (ii)' $\varphi(1) \neq 0 \neq \psi(1)$ and there exists $p \in (0, 1)$ such that $\varphi(t) = \varphi(1)t^p$ and $\psi(t) = \psi(1) = t^{1/p}$ for $t \in \mathbb{R}_+$.

We may now proceed to our main result.

Theorem 6. *Suppose that $\mu(\Sigma) = [0, 1]$ and $\varphi, \psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are bijections satisfying $\varphi(0) = \psi(0) = 0$, inequality (2) and condition (*). Then there exist numbers $p, q > 1$ with $p^{-1} + q^{-1} = 1$ such that $\varphi(t) = \varphi(1)t^p$ and $\psi(t) = \psi(1)t^q$ for $t \in \mathbb{R}_+$.*

Proof. Take any measure space (Y, T, ν) such that $[0, \alpha] \subset \nu(T)$ for some $\alpha > 1$. We wish to apply Theorem 3 to the given measure space (X, Σ, μ) and to (Y, T, ν) . To this end, we shall show that

$$(11) \quad \varphi^{-1} \left\{ \int_X \varphi \left(\int_Y F(x, y) \nu(dy) \right) \mu(dx) \right\} \leq \int_Y \varphi^{-1} \left(\int_X \varphi \circ F(x, y) \mu(dx) \right) \nu(dy)$$

for every $F \in \mathcal{S}_+(X \times Y)$. So, fix any such function F and define $G \in \mathcal{S}_+(X)$ by

$$G(x) = \int_Y F(x, y) \nu(dy) \quad \text{for } x \in X.$$

Let $\chi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function with $\chi(0) = 0$ and such that inequality (2) becomes equality for $(f, g) = (G, \chi \circ G)$. Define also $\Phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by $\Phi(t) = t\chi(t)$ for $t \in \mathbb{R}_+$. Of course, we may assume that $G \neq 0$, since otherwise (11) is trivial. For simplicity, we will treat the ratio $(\Phi \circ G)(x)/G(x)$ as zero for all these $x \in X$ for which $G(x) = 0$. Noticing that $(\Phi \circ G)/G = \chi \circ G$ and using inequality (2), we get:

$$\begin{aligned} \int_X \Phi \left(\int_Y F(x, y) \nu(dy) \right) \mu(dx) &= \int_X \frac{(\Phi \circ G)(x)}{G(x)} \cdot G(x) \mu(dx) \\ &= \int_X \frac{(\Phi \circ G)(x)}{G(x)} \int_Y F(x, y) \nu(dy) \mu(dx) = \int_Y \int_X \frac{(\Phi \circ G)(x)}{G(x)} \cdot F(x, y) \mu(dx) \nu(dy) \\ &\leq \int_Y \mathbb{P}_\varphi(F(\cdot, y)) \mathbb{P}_\psi \left(\frac{\Phi \circ G}{G} \right) \nu(dy) = \mathbb{P}_\psi \left(\frac{\Phi \circ G}{G} \right) \int_Y \mathbb{P}_\varphi(F(\cdot, y)) \nu(dy) \\ &= \mathbb{P}_\psi(\chi \circ G) \int_Y \varphi^{-1} \left(\int_X \varphi \circ F(x, y) \mu(dx) \right) \nu(dy). \end{aligned}$$

We may divide both sides by $\mathbb{P}_\psi(\chi \circ G)$ as $\chi \circ G \neq 0$. By doing so, we obtain nothing else but inequality (11) because by the choice of χ , we have

$$\int_X (\Phi \circ G)(x) \mu(dx) = \int_X G(x) \cdot (\chi \circ G)(x) \mu(dx) = \mathbb{P}_\varphi(G) \mathbb{P}_\psi(\chi \circ G).$$

By virtue of Theorem 3, there is $p \geq 1$ such that $\varphi(t) = \varphi(1)t^p$ for $t \in \mathbb{R}_+$. By symmetry, there is also $q \geq 1$ such that $\psi(t) = \psi(1)t^q$ for $t \in \mathbb{R}_+$. What is left to be proved is the equality $p^{-1} + q^{-1} = 1$.

According to Theorem 2, the pair of functions $\varphi(t) = \varphi(1)t^p$ and $\psi(t) = \psi(1)t^q$ satisfies inequality (2) if and only if the function $F: \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, defined by $F(s, t) = s^{1/p}t^{1/q}$ is concave. This is in turn equivalent to the second Gâteaux differential $d^2F(\mathbf{a})(\mathbf{v}, \mathbf{v})$ being non-positive for every $\mathbf{a} \in (0, \infty)^2$ and $\mathbf{v} \in \mathbb{R}^2$. An easy calculation shows that for every $\mathbf{a} \in (0, \infty)^2$ we have

$$d^2F(\mathbf{a}) = \begin{pmatrix} \frac{1}{p} \left(\frac{1}{p} - 1 \right) & \frac{1}{pq} \\ \frac{1}{pq} & \frac{1}{q} \left(\frac{1}{q} - 1 \right) \end{pmatrix},$$

whence the concavity of F is equivalent to

$$0 \leq \det(d^2F(\mathbf{a})) = \frac{1}{pq} \left(1 - \left(\frac{1}{p} + \frac{1}{q} \right) \right) \iff \frac{1}{p} + \frac{1}{q} \leq 1.$$

Now, suppose we have the strict inequality $p^{-1} + q^{-1} < 1$ and pick any $p' < p$ such that $p'^{-1} + q^{-1} < 1$. Let $\tilde{\varphi}(t) = \varphi(1)t^{p'}$. Then for any non-constant function $f \in \mathcal{S}_+$ we would have $\mathbb{P}_{\tilde{\varphi}}(f) < \mathbb{P}_\varphi(f)$ (see, e.g., [1, §3.11]), so inequality (2) holds true after replacing φ by $\tilde{\varphi}$. However, this would imply that the original inequality is strict for any non-constant map $f \in \mathcal{S}_+$ and any non-zero map $g \in \mathcal{S}_+$, which contradicts condition (*). \square

Now, we wish to derive a counterpart of Theorem 6 for reversed Hölder's inequality (see, e.g., [2, Theorem 13.6]):

Hölder's inequality for $0 < p < 1$. *Let $0 < p < 1$ and q satisfy $p^{-1} + q^{-1} = 1$ (note that $q < 0$) and $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $\psi: (0, \infty) \rightarrow (0, \infty)$ be given as $\varphi(t) = t^p$ and $\psi(t) = t^q$. Then*

$$(12) \quad \int_X fg \, d\mu \geq \mathbb{P}_\varphi(f) \mathbb{P}_\psi(g)$$

for all non-negative functions $f \in L_p(\mu)$ and $g \in L_q(\mu)$, unless $\int_X g^q \, d\mu = 0$.

Note that inequality (12) is claimed only for μ -almost everywhere positive $g \in L_q(\mu)$, and that the function ψ is defined only on $(0, \infty)$. When considering step functions f, g these restrictions may be disregarded, provided we define $\psi(0) = 0$ and replace f by $f \cdot \mathbb{1}_{\text{supp}(g)}$ (where $\text{supp}(g) = \{x \in X: g(x) \neq 0\}$) in inequality (12). Anyway, aiming for a converse theorem to the reversed Hölder inequality, we shall slightly modify our assumptions in comparison to these of Theorem 6. Note also that inequality (12) becomes equality if and

only if the functions g^{-1} and $f^p g^{-q}$ are proportional, so condition $(*)$, after adapting to this new situation, again seems natural:

(**) For every non-zero function $f \in \mathcal{S}_+$ there exists a function $\chi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\chi(0) = 0$ and $\chi(t) > 0$ for $t > 0$ and such that inequality (12) becomes equality for $g = \chi \circ f$. Conversely, for every non-zero $g \in \mathcal{S}_+$ there is a function $\tau: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfying $\tau(0) = 0$ and $\tau(t) > 0$ for $t > 0$ and such that inequality (12) becomes equality for $f = \tau \circ g$.

Theorem 7. *Let $\mu(\Sigma) = [0, 1]$ and $\varphi, \psi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be bijections with $\varphi(0) = \psi(0) = 0$. Suppose that*

$$(13) \quad \int_X f g \, d\mu \geq \mathbb{P}_\varphi(f \cdot \mathbf{1}_{\text{supp}(g)}) \mathbb{P}_\psi(g) \quad \text{for } f, g \in \mathcal{S}_+$$

*and the condition (**) holds true. Then there exist numbers $p \in (0, 1)$ and $q < 0$ with $p^{-1} + q^{-1} = 1$ such that $\varphi(t) = \varphi(1)t^p$ and $\psi(t) = \psi(1)t^q$ for $t \in \mathbb{R}_+$.*

Proof. Take any measure space (Y, T, ν) such that $[0, \alpha] \subset \nu(T)$ for some $\alpha > 1$. Observe that we may safely re-write the calculations from the proof of Theorem 6 using our assumption (13) instead of (2). By doing so, we apply inequality (13) only for the pairs $(f, g) = (F(\cdot, y), \chi \circ G)$, where $y \in Y$. This is legitimate as for every such pair we have $f = f \cdot \mathbf{1}_{\text{supp}(g)}$ because $(\chi \circ G)(x) = 0$ implies $G(x) = 0$, thus $F(x, y) = 0$. Consequently, we obtain the inequality reverse to (11), for every $F \in \mathcal{S}_+(X \times Y)$. In view of Theorem 5, we conclude that for some $p \in (0, 1)$ we have $\varphi(t) = \varphi(1)t^p$ ($t \in \mathbb{R}_+$).

Of course, we cannot repeat the argument above for the function ψ (which is not supposed to be a homeomorphism of \mathbb{R}_+), so let us proceed another way. First, we prove that ψ is continuous on $(0, \infty)$. This will be done with the aid of a ‘convex’ version of Theorem 2 (see [6, Remark 6]), whose proof we repeat below for completeness.

Take any $\lambda \in (0, 1)$ and pick any $A \in \Sigma$ with $\mu(A) = \lambda$ (then $\mu(X \setminus A) = 1 - \lambda$). For arbitrary $t, u, v, w > 0$ let

$$f = \varphi^{-1}(t)\mathbf{1}_A + \varphi^{-1}(u)\mathbf{1}_{X \setminus A} \quad \text{and} \quad g = \varphi^{-1}(v)\mathbf{1}_A + \varphi^{-1}(w)\mathbf{1}_{X \setminus A}.$$

Putting these two functions into (13) gives

$$\lambda F(t, v) + (1 - \lambda)F(u, w) \geq F(\lambda(t, v) + (1 - \lambda)(u, w)),$$

where $F: (0, \infty)^2 \rightarrow (0, \infty)$ is defined by $F(s, t) = \varphi^{-1}(s)\psi^{-1}(t)$. Hence, F is convex on $(0, \infty)^2$, thus it is also continuous (see, [3, Theorem 7.1.1]). But we already know that φ is continuous. Consequently, ψ is continuous (on $(0, \infty)$) as well.

Now, let $q < 0$ be the number satisfying $p^{-1} + q^{-1} = 1$ and denote $\gamma(t) = t^q$ for $t > 0$. Let g be an arbitrary positive step function and let τ be a function from condition (**). Then, by Hölder’s inequality (12), we have

$$\mathbb{P}_\psi(g) = \frac{\int_X (\tau \circ g) \cdot g \, d\mu}{\mathbb{P}_\varphi(\tau \circ g)} \geq \mathbb{P}_\gamma(g).$$

On the other hand, after substituting $f = g^{(-1+q)/p}$ we get equality in (12), that is,

$$\mathbb{P}_\varphi(g^{(-1+q)/p})\mathbb{P}_\psi(g) \leq \int_X g^{(-1+q)/p} \cdot g \, d\mu = \mathbb{P}_\varphi(g^{(-1+q)/p})\mathbb{P}_\gamma(g) \leq \mathbb{P}_\varphi(g^{(-1+q)/p})\mathbb{P}_\psi(g).$$

Consequently, $\mathbb{P}_\psi(g) = \mathbb{P}_\gamma(g)$ for every positive step function g . In particular, putting $g = t\mathbf{1}_A + u\mathbf{1}_{X \setminus A}$, where $\mu(A) = 1/2$ and $t, u > 0$, we get

$$\psi^{-1}\left(\frac{\psi(t) + \psi(u)}{2}\right) = \left(\frac{a^q + b^q}{2}\right)^{1/q} \quad \text{for } t, u > 0.$$

Hence, the map $(0, \infty) \ni t \mapsto \psi(t^{1/q})$ satisfies Jensen's functional equation and, being continuous, is of the form $\psi(t^{1/q}) = at + b$ (see [3, §13.2]). Since ψ maps $(0, \infty)$ onto itself, we conclude that $b = 0$, so $\psi(t) = \psi(1)t^q$ and the proof is completed. \square

3. REMARKS ON THE ASSUMPTIONS

Let us explain that the assumptions upon the measure space (Y, T, ν) in Theorem 3 are essential, not only the inequality $\nu(Y) > 1$, but also the requirement that there is a non-atomic part having measure greater 1, plays an important role.

First, consider the case where $Y = \{y_1, y_2\}$, $\nu\{y_1\} = \nu\{y_2\} = 1$ and (X, Σ, μ) is an arbitrary measure space with $[0, 1] \subset \mu(\Sigma)$. Then, inequality (4) reads as

$$\psi\left\{\int_X \varphi(f(x) + g(x))\mu(dx)\right\} \leq \psi\left(\int_X \varphi(f(x))\mu(dx)\right) + \psi\left(\int_X \varphi(g(x))\mu(dx)\right),$$

where $f(x) = F(x, y_1)$ and $g(x) = F(x, y_2)$. If $\psi = \varphi^{-1}$ then this is nothing else but Minkowski's inequality $\mathbb{P}_\varphi(f + g) \leq \mathbb{P}_\varphi(f) + \mathbb{P}_\varphi(g)$ (for all $f, g \in \mathcal{S}_+(X)$). However, according to Matkowski's result, [4, Theorem 3], such an inequality is equivalent to the fact that the function \mathbf{p}_φ is concave. Hence, in this case inequality (4) obviously does not imply that φ and ψ are power functions.

Now, consider the case where Y and X are probabilistic spaces and let again $\psi = \varphi^{-1}$, where $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing bijection. Recall that the weighted quasi-arithmetic mean with generator φ is given by

$$\mathfrak{M}_\varphi(\mathbf{a}) = \mathfrak{M}_\varphi(\mathbf{a}, \mathbf{q}) = \varphi^{-1}\left\{\sum_{j=1}^n q_j \varphi(a_j)\right\},$$

for any finite sequence $\mathbf{a} = (a_1, \dots, a_n)$ of non-negative numbers, and any sequence $\mathbf{q} = (q_1, \dots, q_n)$ of weights, *i.e.*, non-negative numbers summing up to 1. There is a classical result (see [1, Theorem 106(i)]) which says that whenever

- (i) φ is four times continuously differentiable on $(0, \infty)$ and
- (ii) the functions: φ , φ' and φ'' are positive on $(0, \infty)$,

then the inequality

$$(14) \quad \mathfrak{M}_\varphi\left(\frac{\mathbf{a} + \mathbf{b}}{2}\right) \leq \frac{1}{2}(\mathfrak{M}_\varphi(\mathbf{a}) + \mathfrak{M}_\varphi(\mathbf{b}))$$

holds true for all non-negative sequences \mathbf{a} , \mathbf{b} , and all non-negative weights if and only if

(iii) the function φ'/φ'' is concave.

Suppose that φ has all the properties (i)-(iii). By routine arguments, we may then conclude that inequality (14) holds true in the integral form, that is, when we replace the arithmetic mean at the both sides of (14) by integrating with respect to some probabilistic measure. Of course, we may do the same thing with the weighted mean in the definition of \mathfrak{M}_φ . Hence, inequality (14) takes the form

$$\varphi^{-1} \left\{ \int_X \varphi \left(\int_Y F(x, y) \nu(dy) \right) \mu(dx) \right\} \leq \int_Y \varphi^{-1} \left(\int_X \varphi \circ F(x, y) \mu(dx) \right) \nu(dy)$$

(for all $F \in \mathcal{S}_+(X \times Y)$), which is nothing else but inequality (4) with $\psi = \varphi^{-1}$. However, conditions (i)-(iii) obviously do not imply that φ is a power function.

The same remarks, with obvious changes, are valid for Theorem 5.

Concerning the assumption $\mu(\Sigma) = [0, 1]$ let us pose the following question: Under what weaker assumptions upon the measure space (X, Σ, μ) the assertions in Theorems 6 and 7 remain true?

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